Imperial College London

Lecture 7

More on Laplace Transform

(Lathi 4.3 – 4.4)

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Initial & Final Value Theorems

- How to find the initial and final values of a function x(t) if we know its Laplace Transform X(s)? (t ➔ 0^+, and t ➔ ∞)

Initial Value Theorem

$$\lim_{t\to 0} x(t) = x(0^+) = \lim_{s\to\infty} sX(s)$$

- Conditions:
  - Laplace transforms of x(t) and dx/dt exist.
  - X(s) numerator power (M) is less than denominator power (N), i.e. M<N.

Final Value Theorem

$$\lim_{t\to\infty} x(t) = x(\infty) = \lim_{s\to 0} sX(s)$$

- Conditions:
  - Laplace transforms of x(t) and dx/dt exist.
  - sX(s) poles are all on the Left Plane or origin.

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Example

- Find the initial and final values of y(t) if Y(s) is given by:

$$Y(s) = \frac{10(2s+3)}{s(s^2+2s+5)}$$

$$\text{initial value: } y(0+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \frac{10(2s+3)}{(s^2+2s+5)} = 0$$

$$\text{final value: } y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{10(2s+3)}{(s^2+2s+5)} = 6$$

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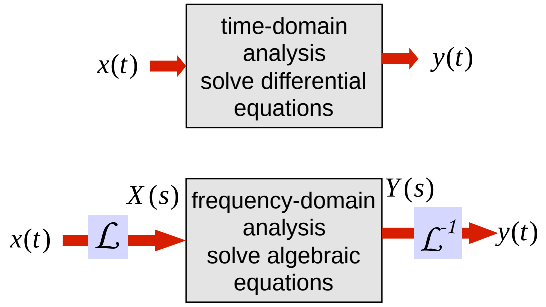
Laplace Transform for Solving Differential Equations

- Remember the time-differentiation property of Laplace Transform

$$\frac{d^k y}{dt^k} \Leftrightarrow s^k Y(s)$$

- Exploit this to solve differential equation as algebraic equations:



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Example (1)

- Solve the following second-order linear differential equation:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + x(t)$$

- Given that $y(0^-) = 2,\ \dot{y}(0^-) = 1$ and input $x(t) = e^{-4t}u(t)$.

Time Domain	Laplace (Frequency) Domain
$\frac{dy}{dt}$	$sY(s) - y(0^-) = sY(s) - 2$
$\frac{d^2y}{dt^2}$	$s^2Y(s) - sy(0^-) - \dot{y}(0^-) = s^2Y(s) - 2s - 1$
$x(t) = e^{-4t}u(t)$	$X(s) = \frac{1}{s+4}$
$\frac{dx}{dt}$	$sX(s) - x(0^-) = \frac{s}{s+4} - 0 = \frac{s}{s+4}$

Example (2)

Time Domain

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + x(t)$$

$$y(t) = \left(\tfrac{13}{2}e^{-2t} - 3e^{-3t} - \tfrac{3}{2}e^{-4t}\right)u(t)$$

Laplace (Frequency) Domain

$$[s^2Y(s) - 2s - 1] + 5[sY(s) - 2] + 6Y(s) = \frac{s}{s+4} + \frac{1}{s+4}$$

$$(s^2 + 5s + 6)Y(s) - (2s + 11) = \frac{s+1}{s+4}$$

$$(s^2 + 5s + 6)Y(s) = \frac{2s^2 + 20s + 45}{s+4}$$

$$Y(s) = \frac{2s^2 + 20s + 45}{(s+2)(s+3)(s+4)}$$

$$Y(s) = \frac{13/2}{s+2} - \frac{3}{s+3} - \frac{3/2}{s+4}$$

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Zero-input & Zero-state Responses

- Let's think about where the terms come from: $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + x(t)$

$$(s^2 + 5s + 6)Y(s) - \underbrace{(2s + 11)}_{\text{Initial condition term}} = \underbrace{\frac{s+1}{s+4}}_{\text{Input term}}$$

$$Y(s) = \underbrace{\frac{2s+11}{s^2+5s+6}}_{\text{zero-input component}} + \underbrace{\frac{s+1}{(s+4)(s^2+5s+6)}}_{\text{zero-state component}}$$

$$= \left[\frac{7}{s+2} - \frac{5}{s+3}\right] + \left[\frac{-1/2}{s+2} + \frac{2}{s+3} - \frac{3/2}{s+4}\right]$$

$$y(t) = \underbrace{\left(7e^{-2t} - 5e^{-3t}\right)u(t)}_{\text{zero-input response}} + \underbrace{\left(-\tfrac{1}{2}e^{-2t} + 2e^{-3t} - \tfrac{3}{2}e^{-4t}\right)u(t)}_{\text{zero-state response}}$$

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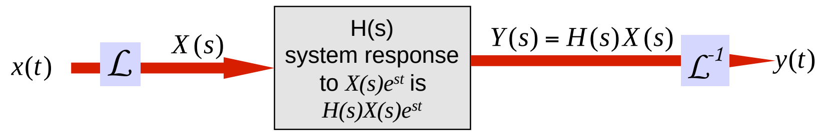
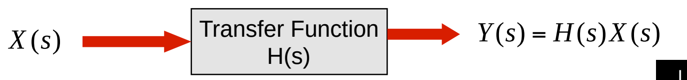
Laplace Tranform and Transfer Function

- Let's express input x(t) as a linear combination of exponentials e^{st}:

$$x(t) = \sum_{i=1}^{K} X(s_i)e^{s_i t}$$

- H(s) can be regarded as the system's response to each of the exponential components, in such a way that the output y(t) is: $y(t) = \sum_{i=1}^{K} X(s_i)H(s_i)e^{s_i t}$
- Therefore, we get $Y(s) = H(s)\,X(s)$





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Transfer Function Examples

$x(t)$ → Delay by T sec → $y(t) = x(t-T)$

$X(s)$ ⇕ $H(s) = Y(s)/X(s) = e^{-sT}$ ⇕ $Y(s) = X(s)e^{-sT}$

Shifting Property

$x(t)$ → Differentiator d/dt → $y(t) = dx/dt$

$X(s)$ ⇕ $H(s) = s$ ⇕ $Y(s) = sX(s)$

Differentiation Property

$x(t)$ → Integrator → $y(t) = \int_0^t x(\tau)d\tau$

$X(s)$ ⇕ $H(s) = \frac{1}{s}$ ⇕ $Y(s) = \frac{1}{s}X(s)$

Integration Property

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Initial conditions in systems (1)

- In circuits, initial conditions may not be zero. For example, capacitors may be charged; inductors may have an initial current.
- How should these be represented in the Laplace (frequency) domain?
- Consider a capacitor C with an initial voltage v(0⁻): $i(t) = C\frac{dv}{dt}$
- Now take Laplace transform on both sides:

$$I(s) = C[sV(s) - v(0^-)]$$

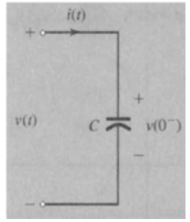
- Rearrange this to give:

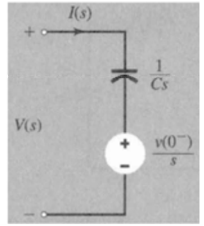
$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0^-)}{s}$$

voltage across charged capacitor

voltage across capacitor with no charge

effect of the initial charge = voltage source





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Initial conditions in systems (2)

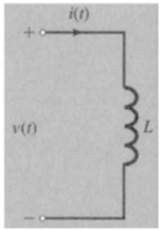
- Similarly, consider an inductor L with an initial current i(0⁻):
- Consider a capacitor C with an initial voltage v(0⁻): $v(t) = L\frac{di}{dt}$
- Now take Laplace transform on both sides:
- Rearrange this to give:

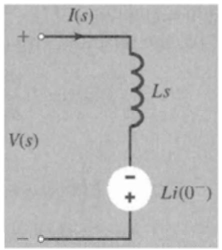
$$V(s) = L[sI(s) - i(0^-)]$$
$$= LsI(s) - Li(0^-)$$

voltage across inductor

voltage across inductor with no initial current

effect of the initial current = voltage source





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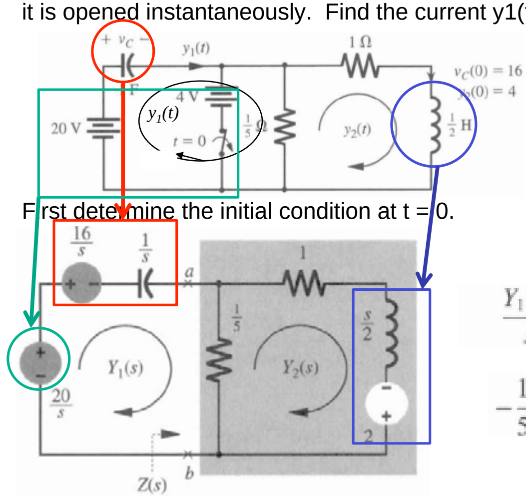
Solving Transient Behaviour in circuits – Example 1(1)

- The switch in the circuit here is in closed position for a long time before t=0, when it is opened instantaneously. Find the current y1(t) and y2(t) for t>0.



- First determine the initial condition at t = 0.

$$\frac{Y_1(s)}{s} + \frac{1}{5}[Y_1(s) - Y_2(s)] = \frac{4}{s}$$

$$-\frac{1}{5}Y_1(s) + \frac{6}{5}Y_2(s) + \frac{s}{2}Y_2(s) = 2$$

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Example 1 (2)

- From this we can rewrite as in matrix form: $\begin{bmatrix} \frac{1}{s}+\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{6}{5}+\frac{s}{2} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{4}{s} \\ 2 \end{bmatrix}$
- We need to solve for $Y_1(s)$ and $Y_2(s)$.
- We do this by applying Cramer's rule, which is:
- Given $\mathbf{A}z = c$, where $\mathbf{A}$ is a square matrix, z and c are column vectors, the vector z can be solve by: $z_i = \dfrac{\det(A_i)}{\det(A)}$

where A_i is the matrix A with its i[th] column replaced by column vector c.

$$ax + by = e$$
$$cx + dy = f$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - bf}{ad - bc} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ec}{ad - bc}$$

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Example 1(3)

- We readily obtain:

$$\det(A) = \det \begin{bmatrix} \frac{1}{s}+\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{6}{5}+\frac{s}{2} \end{bmatrix} = \frac{1}{10s}(s^2 + 7s + 12)$$

and therefore:

$$Y_1(s) = \frac{\det \begin{bmatrix} \frac{4}{s} & -\frac{1}{5} \\ 2 & \frac{6}{5}+\frac{s}{2} \end{bmatrix}}{\det(A)} = \frac{24(s+2)}{s^2 + 7s + 12} = \frac{-24}{s+3} + \frac{48}{s+4}$$

- Inverse Laplace gives us: $y_1(t) = (-24e^{-3t} + 48e^{-4t})u(t)$
- Similarly we obtain: $Y_2(s) = \dfrac{4(s+7)}{s^2+7s+12} = \dfrac{16}{s+3} - \dfrac{12}{s+4}$
- Therefore: $y_2(t) = (16e^{-3t} - 12e^{-4t})u(t)$

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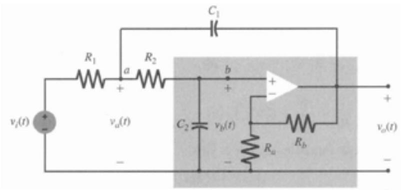
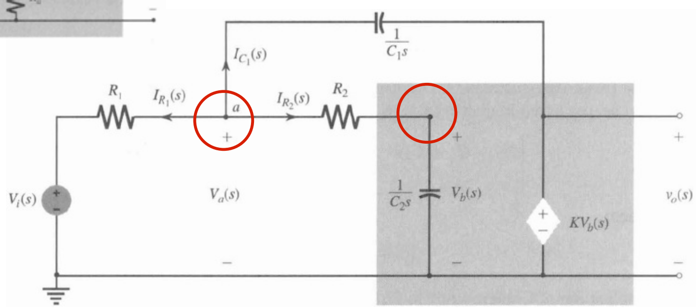
Solving Transient Behaviour in circuits – Example 2(1)

- Find the transfer function $H(s)$ relating the output $v_o(t)$ to the input voltage $v_i(t)$ for the Sallen and Key filter shown below. Assume that initial condition is zero.



Step 1: Form equivalent circuit
Step 2: Pick "variables" – nodal voltages at a and b



$$K = 1 + \frac{R_b}{R_a}$$

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Solving Transient Behaviour in circuits – Example 2(2)

Step 3: Sum current at node a

$$\frac{V_a(s) - V_i(s)}{R_1} + \frac{V_a(s) - V_b(s)}{R_2} + [V_a(s) - KV_b(s)]C_1 s = 0$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + C_1 s\right) V_a(s) - \left(\frac{1}{R_2} + KC_1 s\right) V_b(s) = \frac{1}{R_1} V_i(s)$$

Step 4: Sum current at node b

$$\frac{V_b(s) - V_a(s)}{R_2} + C_2 s V_b(s) = 0$$

$$-\frac{1}{R_2} V_a(s) + \left(\frac{1}{R_2} + C_2 s\right) V_b(s) = 0$$

Step 5: Put in matrix form

$$\begin{bmatrix} G_1 + G_2 + C_1 s & -(G_2 + KC_1 s) \\ -G_2 & (G_2 + C_2 s) \end{bmatrix} \begin{bmatrix} V_a(s) \\ V_b(s) \end{bmatrix} = \begin{bmatrix} G_1 V_i(s) \\ 0 \end{bmatrix}$$

$$G_1 = \frac{1}{R_1} \qquad G_2 = \frac{1}{R_2} \qquad K = 1 + \frac{R_b}{R_a}$$

Solving Transient Behaviour in circuits – Example 2(3)

Step 6: Apply Cramer's rule

$$\frac{V_b(s)}{V_i(s)} = \frac{G_1 G_2}{C_1 C_2 s^2 + [G_1 C_2 + G_2 C_2 + G_2 C_1 (1-K)]s + G_1 G_2}$$
$$= \frac{\omega_0{}^2}{s^2 + 2\alpha s + \omega_0{}^2}$$

$$K = 1 + \frac{R_b}{R_a} \quad \text{and} \quad \omega_0{}^2 = \frac{G_1 G_2}{C_1 C_2} = \frac{1}{R_1 R_2 C_1 C_2}$$
$$2\alpha = \frac{G_1 C_2 + G_2 C_2 + G_2 C_1 (1-K)}{C_1 C_2} = \frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} + \frac{1}{R_2 C_2}(1-K)$$

Step 7: Derive H(s)

$$V_o(s) = K V_b(s)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = K \frac{V_b(s)}{V_i(s)} = \frac{K \omega_0{}^2}{s^2 + 2\alpha s + \omega_0{}^2}$$

Relating this lecture to other courses

- You have done much of the circuit analysis in your first year, but Laplace transform provides much more elegant method in find solutions to BOTH transient and steady state condition of circuits.
- You have done Sallen-and-Key filter in your 2nd year analogue circuits course. Here we derive the transfer function from first principle, using only tools you know about.
- The treatment provided in this lecture also enhances what you have been learning in your 2nd year control course.